Last time we introduced map $Z(L;\lambda_1,\dots,\lambda_m)$ as a composition of maps Z_j, $0 \le j \le s-1$

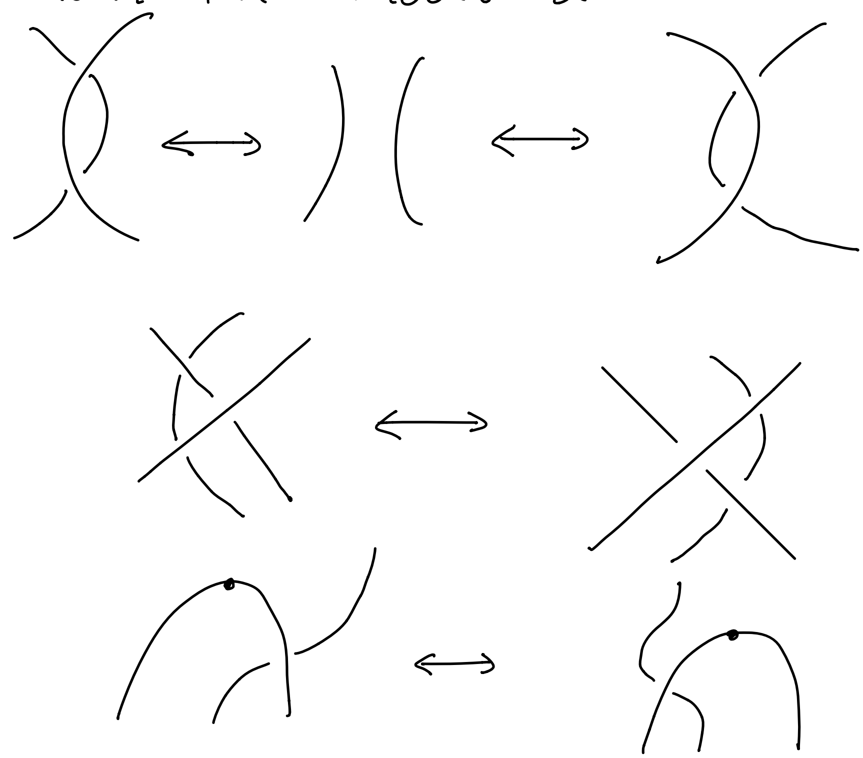
Z is a $\mathbb{C}$-number as it is a linear map

$$Z : \underset{\mathbb{C}}{\underset{\parallel}{V(t_0)}} \longrightarrow \underset{\mathbb{C}}{\underset{\parallel}{V(t_s)}}$$

Now flatness of KZ connection gives:

__Lemma 1:__

$Z(L;\lambda_1,\dots,\lambda_m)$ is invariant under the local horizontal moves below

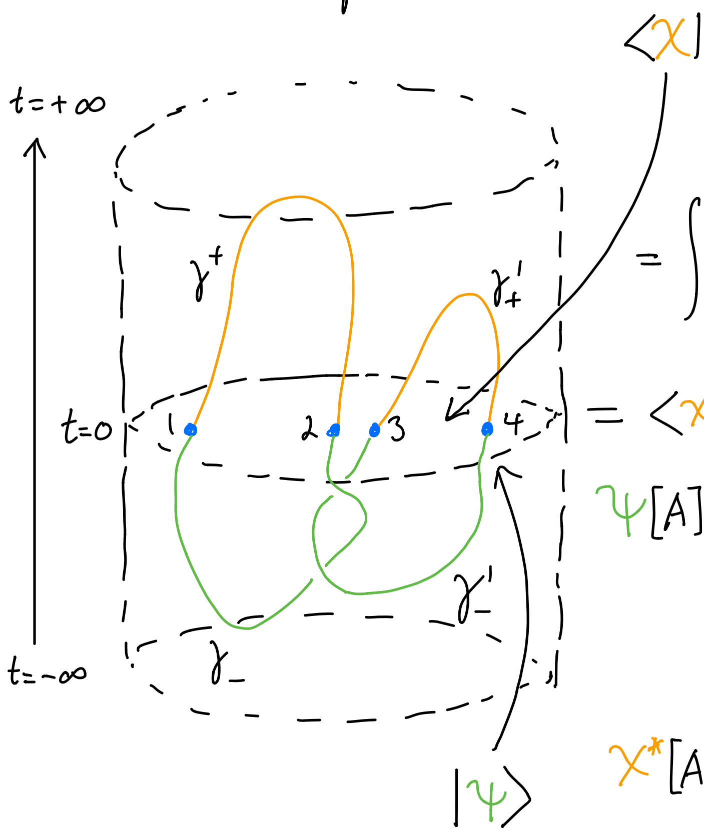


These are also called "Reidemeister moves"

Next, we want to interpret Z in the context of Chern-Simons theory

$\longrightarrow$ restrict to 4 conformal primaries in spin λ representation of $SU(2)$ at each time slice

$\longrightarrow$ vacuum to vacuum amplitude can be expressed as



$$\langle 0|0\rangle_{\gamma_1,\lambda_1;\gamma_2,\lambda_2} = \int \mathcal{D}a\, e^{2\pi i\, cs(a)} W_{\gamma,\lambda}[a] W_{\gamma',\lambda'}[a]$$

$$\parallel$$

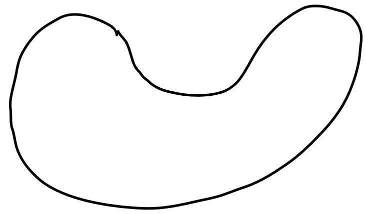
$$= \langle \chi|\psi\rangle \quad \text{where}$$

$$\psi[A] = \int\limits_{a(\vec{x},0)=A(\vec{x})} \mathcal{D}a(\vec{x},t)\, W_{\gamma_-,\lambda}[a] W_{\gamma'_-,\lambda'}[a] \times \exp\left(\int_{-\infty}^{0} dt \int d^2\vec{x}\, \mathcal{L}_{CS}\right)$$

$$\chi^*[A] = \int\limits_{a(\vec{x},0)=A(\vec{x})} \mathcal{D}a(\vec{x},t)\, W_{\gamma_+,\lambda}[a] W_{\gamma'_+,\lambda'}[a] \times \exp\left(\int_{0}^{\infty} dt \int d^2\vec{x}\, \mathcal{L}_{CS}\right)$$

$$\boxed{|\psi\rangle = \rho(\sigma_2^{\,2})\,|\chi\rangle}$$

Next, define

$K_o =$ 

as the unique unknot with two minimal and two maximal points and set

$$d(\lambda) = Z(K_o ; \lambda)^{-1}$$

Define $\mu(j) = \#$ maximal points in L_j

$\rightarrow J(L; \lambda_1, \ldots, \lambda_m) = d(\lambda_1)^{\mu(1)} \cdots d(\lambda_m)^{\mu(m)} Z(L; \lambda_1, \ldots, \lambda_m)$

<u>Theorem 1:</u>

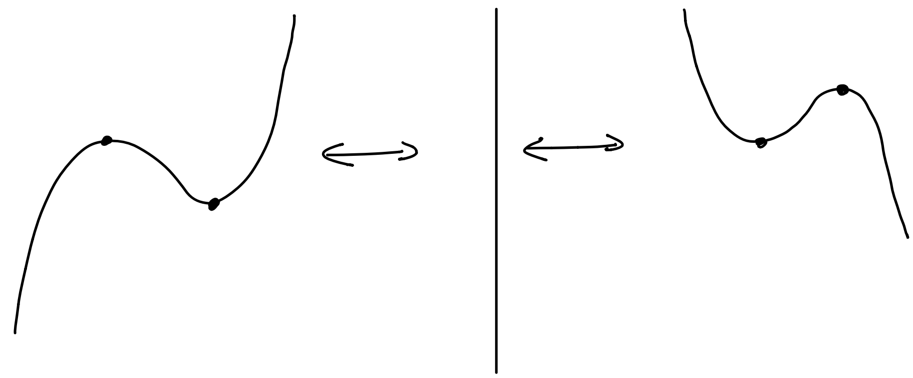
$J(L; \lambda_1, \ldots, \lambda_m)$ is an invariant of a colored oriented framed link.

<u>Proof:</u>

Suppose $L' = L_1' \cup \cdots \cup L_m'$ is an equivalent link, i.e. $\exists$ orientation preserving homeomorphism h of S^3 s.t. $h(L_j) = L_j'$, $1 \le j \le m$

Have to show: $J(L) = J(L')$

$L' \longleftrightarrow L$ if and only if L' is obtained from L through a sequence of Reidemeister moves and a cancellation of two critical points

(*) 

Lemma 1 shows invariance of Z under Reidemeister moves

$\longrightarrow$ have to show invariance under (*)

$L' = L + K_0$

$\Rightarrow Z(L'; \lambda_1, \dots, \lambda_m) = Z(K_0; \lambda_j) Z(L; \lambda_1, \dots, \lambda_m)$

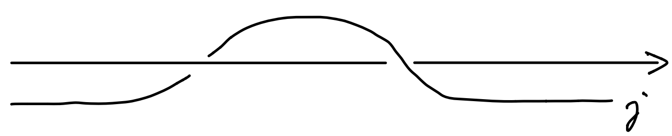
$\longrightarrow J(L_i, \lambda_1, \dots, \lambda_m)$ is invariant under a cancellation of critical points

<u>Proposition 1:</u>

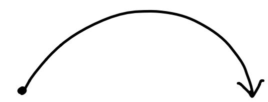
Let L' be a link obtained from $L = \bigcup_{j=1}^{m} L_j$ by increasing the framing of the component L_j by $1 \longrightarrow J(L'; \lambda_1, \dots, \lambda_m) = \exp 2\pi\sqrt{-1}\, \Delta_{\lambda_j} J(L; \lambda_1, \dots, \lambda_m)$

Proof:

Increase of framing by 1 means:



cross-section:



or $z_j \longmapsto e^{\pi i} z_j$

We know that under $w_j = \dfrac{a z_j + b}{c z_j + d}$

conformal blocks $\Psi_0(z_1, \dots, z_n)$ transform as

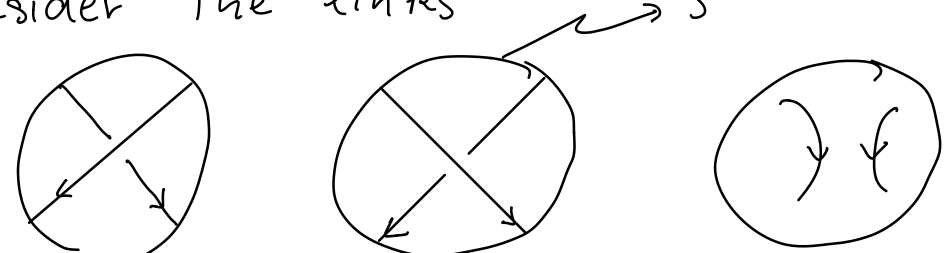
$$\Psi_0(z_1, \dots, z_n) = \prod_{j=1}^{n} (c z_j + d)^{-2\Delta_{\lambda_j}} \Psi_0(w_1, \dots, w_n)$$

or for $z_j \longmapsto \alpha z_j$: $\Psi_0(z_1, \dots, z_n) = \alpha^{2\Delta_{\lambda_j}} \Psi_0(w_1, \dots, w_n)$

set $\alpha = e^{\pi\sqrt{-1}}$ $\square$

Notation: In the case $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 1$,
we write J_L for $J(L; \lambda, \dots, \lambda)$

Consider the links S^3



L_+ L_- L_0

and identical outside S^3.

Proposition 2:

set $q^{1/m} = \exp\left(\frac{2\pi\sqrt{-1}}{m(K+2)}\right)$

The link invariant J_L satisfies the skein relation

$$q^{1/4} J_{L_+} - q^{-1/4} J_{L_-} = \left(q^{1/2} - q^{-1/2}\right) J_{L_0} \quad (*)$$

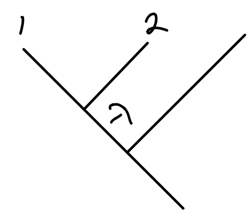
Proof:

We have seen that the monodromy matrix $\rho(\sigma_1)$ acts on conformal blocks as

$$\rho(\sigma_1) = P_{12} \exp\left(\pi\sqrt{-1}\, \Omega_{12}/k\right), \quad k = K+2$$

The matrix Ω_{12}/k is diagonalized with eigenvalues

$$\Delta_\lambda - \Delta_{\lambda_1} - \Delta_{\lambda_2}$$



Since $\lambda_1 = \lambda_2 = 1$ we get from Clebsch-Gordan rule $\lambda = 0, 2$

Recall $\Delta_{2j} = \frac{j(j+1)}{K+2} \Rightarrow \Delta_0 = 0,\ \Delta_1 = \frac{3/4}{K+2},$

$\Delta_2 = \frac{2}{K+2}$

Set $G_i = \rho(\sigma_i)$. Then we have

$$\left(G_i + q^{-3/4}\right)\left(G_i - q^{1/4}\right)$$

$$= G_i^2 - G_i q^{1/4} + q^{-3/4} G_i - q^{-1/2} = 0$$

This is equivalent to

$$q^{1/4} G_i - q^{-1/4} G_i^{-1} = \left(q^{1/2} - q^{-1/2}\right) \mathbb{1}$$

$$\langle 0 | \rho(L_-)\, \rho(\sigma_i)\, \rho(L_+) | 0 \rangle$$

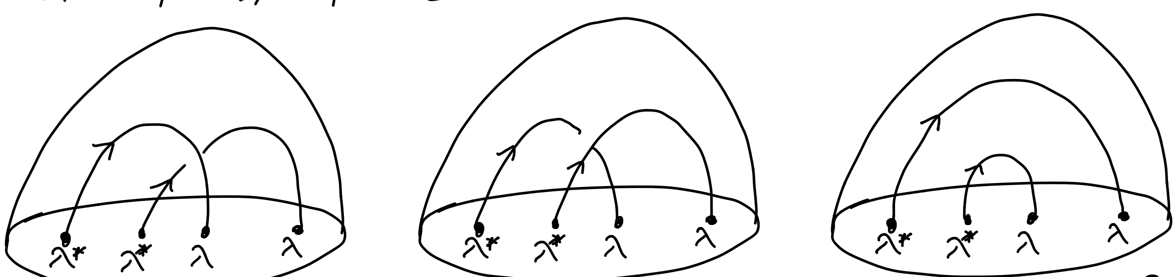
$$= \sum_{\lambda,\mu} \langle 0 | \rho(L_-) | \chi_\lambda \rangle \langle \chi_\lambda | \rho(\sigma_i) | \chi_\mu \rangle \langle \chi_\mu | \rho(L_+) | 0 \rangle$$

$$= \sum_{\lambda} \langle 0 | \rho(L_-) | \chi_\lambda \rangle \langle \chi_\lambda | \rho(\sigma_i) | \chi_\lambda \rangle \langle \chi_\lambda | \rho(L_+) | 0 \rangle$$

$\square$

<u>Interpretation:</u>

Consider the space of conformal blocks $\mathcal{H}$ on $\mathbb{CP}^1$ with four points and highest weights $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. In our case: $\lambda, \lambda, \lambda^*, \lambda^*$



$L_+ \to v_+ \in \mathcal{H}$ $\quad$ $L_- \to v_- \in \mathcal{H}$ $\quad$ $L_0 \to v_0 \in \mathcal{H}$

There exists a vector $\omega \in \mathcal{H}^*$ such that

$$\mathcal{J}_{L_+} = \langle \omega | v_+ \rangle, \quad \mathcal{J}_{L_-} = \langle \omega | v_- \rangle, \quad \mathcal{J}_{L_0} = \langle \omega, v_0 \rangle$$

where $\langle \,|\, \rangle : \mathcal{H}^* \times \mathcal{H} \longrightarrow \mathbb{C}$ is natural pairing

$\dim \mathcal{H} = 2$:

- if primaries 1 and 2 fuse to $\lambda = 0$, then 3 and 4 must as well
- if primaries 1 and 2 fuse to $\lambda = 2$, then 3 and 4 must as well.

$$\Rightarrow \alpha v_+ + \beta v_- + \gamma v_0 = 0$$

$$\rightarrow \alpha \mathcal{J}_+ + \beta \mathcal{J}_{L_-} + \gamma \mathcal{J}_{L_0} = 0$$

α, β and γ are determined by (*).

<u>Note</u>: Our invariant $\mathcal{J}_L$ depends on the framing. To cure this, denote by $w(L)$ the "writhe" of L (# positive crossings − # negative crossings), and set $P_L = d(1)^{-1} \exp\left(-2\pi\sqrt{-1}\,\Delta_1 w(L)\right)\mathcal{J}_L$